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2004 J. Phys. A: Math. Gen. 37 9251

(http://iopscience.iop.org/0305-4470/37/39/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 9251–9260 PII: S0305-4470(04)78032-4

Entanglement measure for general pure multipartite quantum states

Hoshang Heydari and Gunnar Björk

Department of Microelectronics and Information Technology, Royal Institute of Technology (KTH), Electrum 229, SE-164 40 Kista, Sweden

E-mail: hoshang@imit.kth.se

Received 18 March 2004, in final form 28 June 2004
Published 15 September 2004
Online at stacks.iop.org/JPhysA/37/9251
doi:10.1088/0305-4470/37/39/012

Abstract

We propose an explicit formula for a measure of entanglement of pure multipartite quantum states. We discuss the mathematical structure of the measure and give a brief explanation of its physical motivation. We apply the measure on some pure, tripartite, qubit states and demonstrate that, in general, the entanglement can depend on what actions are performed on the various subsystems, and specifically if the parties in possession of the subsystems cooperate or not. We also give some simple but illustrative examples of the entanglement of four-qubit and m-qubit states.

PACS numbers: 03.67.Mn, 42.50.Dv, 42.50.Hz, 42.65.Ky

1. Introduction

One of the unsolved problems of modern quantum theory is the quantification of multipartite-state entanglement [1, 2]. This is a task that is directly linked to mathematics, such as linear algebra, geometry and functional analysis. The definition of separability and of entanglement of a multipartite state was introduced in [3], following the definition for bipartite states, given in 1989 by Werner [4]. Eventually, quantitative measures, such as the entanglement of formation and concurrence were formulated for bipartite systems [5]. In recent years, there have been attempts to find entanglement measures for qubit–qudit states [6–9] and for multipartite states, i.e., in [10–15]. To exemplify: upper and lower bounds for the quantum relative-entropy of entanglement of a multipartite system, in terms of the bipartite entanglements of formation, distillation and quantum entropy of various subsystems, are derived in [10]. Measures based on the Schmidt rank are proposed in [13], and on local unitary and filtering processes in [14]. Generalized concurrence for pure, multipartite states in arbitrary dimensions has been derived in [7], based on invariants under local unitary transformation. Another generalization of the concurrence for arbitrary, pure, multipartite states is based on the representation of generators

of the corresponding special orthogonal groups [16]. This measure is defined as the norm of certain concurrence vectors. Unfortunately, the measure cannot be used to completely quantify a general, pure, multipartite state. A rather different entanglement measure for general, pure, multipartite states, based on a generalized Schmidt-decomposition, has been proposed in [17]. Furthermore, in [18], a very useful tool to detect entanglement, called entanglement witness, is generalized to multipartite states. The tool is a consequence of the Hahn–Banach theorem which states that for any convex, compact, vector set $\mathcal{S}$, if $\rho \notin \mathcal{S}$, there exists a hyperplane that separates ρ from $\mathcal{S}$. However, to find such an operator, even in the case of a tripartite state, is a formidable task. Nonetheless, quite impressively, Acín *et al* managed to construct a witness operator for a class of mixed tripartite states [19].

In this paper, we propose another measure of entanglement for arbitrary, pure, multipartite states, inspired by the work on the role of the relative phase between subsystems in describing entanglement discussed in [20]. Our method is based on the joint relative-phase properties of a multipartite quantum system $\mathcal{Q} = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$ in a Hilbert space $\mathcal{H}_\mathcal{Q} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$ expressed by a positive operator value measure (POVM) $\Delta_\mathcal{Q}$ on $\mathcal{H}_\mathcal{Q}$. The POVM is constructed by taking the m-fold tensor product of the subsystems' corresponding POVMs. We have already derived and discussed, in detail, our measure of entanglement for bipartite states in [21, 22], so here we will only discuss examples of multipartite entangled states.

2. Entanglement from a relative-phase POVM

A general and symmetric POVM in a single N_u-dimensional Hilbert space $\mathcal{H}_{\mathcal{Q}_u}$ is given by

$$\hat{\Delta}_{\mathcal{Q}_u} = \sum_{l_u=1}^{N_u}\sum_{k_u=1}^{N_u} \mathrm{e}^{\mathrm{i}\varphi_{k_u,l_u}} |k_u\rangle\langle l_u|, \tag{1}$$

where $|k_u\rangle$ (and $|l_u\rangle$) are the basis vectors in $\mathcal{H}_{\mathcal{Q}_u}$ and

$$\varphi_{k_u,l_u} = -\varphi_{l_u,k_u}(1-\delta_{k_u l_u}). \tag{2}$$

The POVM is a function of the $N_u(N_u-1)/2$ relative phases $\big(\varphi_{1_u,2_u},\ldots,\varphi_{1_u,N_u}, \varphi_{2_u,3_u},\ldots,\varphi_{N_u-1,N_u}\big)$.

It is now possible to form a POVM of a multipartite system by simply forming the tensor product

$$\hat{\Delta}_\mathcal{Q}\big(\varphi_{\mathcal{Q}_1;k_1,l_1},\ldots,\varphi_{\mathcal{Q}_m;k_m,l_m}\big) = \hat{\Delta}_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1;k_1,l_1}\big)\otimes\cdots\otimes\hat{\Delta}_{\mathcal{Q}_m}\big(\varphi_{\mathcal{Q}_m;k_m,l_m}\big), \tag{3}$$

where, e.g., $\varphi_{\mathcal{Q}_1;k_1,l_1}$ is the set of POVM relative phases associated with subsystems $\mathcal{Q}_1$, for all $k_1, l_1 = 1, 2, \ldots, N_1$, where we need to only consider when $l_1 > k_1$ due to (2). We can now recast this POVM, expressed in local properties, in terms of the relative-phase sums and differences $\phi_{k_1,l_1,\ldots,k_m,l_m} = \sum_{u=1}^m \varphi_{k_u,l_u}$. Note that if, e.g., $l_v = k_v$, then the term φ_{k_v,k_v} vanishes from the sum due to (2). From $\hat{\Delta}_\mathcal{Q}$ we can express the probability $\mathcal{P}$ that a measurement process results in the particular combination of joint relative phases $\phi_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}},\ldots,\phi_{k_1^{(M)},l_1^{(M)},\ldots,k_m^{(M)},l_m^{(M)}}$ as

$$\mathcal{P}\big(\phi_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}},\ldots,\phi_{k_1^{(M)},l_1^{(M)},\ldots,k_m^{(M)},l_m^{(M)}}\big) = \mathrm{Tr}\big(\hat{\rho}\hat{\Delta}\big(\phi_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}},\ldots,\phi_{k_1^{(M)},l_1^{(M)},\ldots,k_m^{(M)},l_m^{(M)}}\big)\big), \tag{4}$$

where $\hat{\rho}$ is the state density operator acting on the composite Hilbert space $\mathcal{H}_\mathcal{Q}$, and the argument of $\mathcal{P}$ contains $\Pi_{u=1}^m N_u(N_u-1)/2 = M$ linearly independent relative-phase sums.

Next, we define an (unnormalized) marginal probability of obtaining the particular relative-phase outcome $\phi_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}}$:

$$\gamma_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}} = \left| \int_{2\pi} \mathrm{d}\phi_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}} \, \mathrm{e}^{-\mathrm{i}\phi_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}}} \times \mathcal{P}\left(\phi_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}}, \ldots, \phi_{k_1^{(M)},l_1^{(M)},\ldots,k_m^{(M)},l_m^{(M)}}\right) \right|, \tag{5}$$

where $\mathcal{P}$ must be expressed in the relative-phase sum and difference parameter $\phi_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}}$, but the particular choice of the remaining $M-1$ linearly independent relative-phase sum and difference parameters is inconsequential for the absolute value of the integral. The latter fact is physically motivated by the fact that $\gamma_{k_1^{(1)},l_1^{(1)},\ldots,k_m^{(1)},l_m^{(1)}}$ is an (unnormalized) statistical marginal probability. The marginal probabilities $\gamma_{k_u,l_u,\ldots,k_v,l_v}$ (where, here, and in the following, we will omit the superscript on the indices) are proportional to the Fourier components of the joint relative-phase distribution. Now, let us introduce the following index operator to connect the notation using the subsystem indices, and the notation using a joint-system index running from 1 to $N_1 N_2 \cdots N_m$:

$$\begin{aligned} \Pi(k_1,l_1,k_2,l_2,\ldots,k_m,l_m) &= (k_1-1)N_2\cdots N_m + (k_2-1)N_3\cdots N_m \\ &\quad + \cdots + (k_{m-1}-1)N_m + k_m, \\ (l_1-1)N_2\cdots N_m + (l_2-1)N_3\cdots N_m + \cdots + (l_{m-1}-1)N_m + l_m. \end{aligned} \tag{6}$$

Note that the index operator generates two indices based on the sets $\{k_u\}$ and $\{l_u\}$, respectively. Evaluating the Fourier components, one finds, not surprisingly, that $\gamma_{k_1,l_1,\ldots,k_m,l_m} = 2\pi|\rho_{\Pi(k_1,l_1,\ldots,k_m,l_m)}|$. That is, to each relative-phase sum and difference there is an associated joint-system density matrix coefficient, where the relative-phase sums contribute by a positive term and the relative-phase differences with a negative term. We now define an index permutation operator P_j operating on any function $f(k_1,l_1,\ldots,k_m,l_m)$ by

$$\begin{aligned} \mathrm{P}_j f(k_1,l_1,\ldots,k_j,l_j,\ldots,k_m,l_m) &= f(k_1,l_1,\ldots,k_j,l_j,\ldots,k_m,l_m) \\ &\quad - f(k_1,l_1,\ldots,l_j,k_j,\ldots,k_m,l_m). \end{aligned} \tag{7}$$

Using this operator, we can generalize our earlier results for bipartite systems [21, 22]. We form an entanglement function by summing the absolute difference between pairwise relative-phase sums and differences. The motivation for doing so can be explained by the following (not quite complete) analogy. Suppose we have the EPR-state $(|0,0\rangle + |1,1\rangle)/\sqrt{2}$ and that we make measurements on each subsystem yielding the number zero for state $|0\rangle$ and unity for the state $|1\rangle$. If we then form the number sum and the number difference between the two subsystems, we find that the measured number sum is uncorrelated, randomly alternating between $0+0=0$ and $1+1=2$. The number difference, in contrast, is correlated, always equalling zero. The situation is the reverse for the state $(|0,1\rangle + |1,0\rangle)/\sqrt{2}$. By subtracting the (absolute values of the) properly normalized number-difference and number-sum correlations and taking the absolute value of the result, we get a positive number for either state. However, for an equal and real superposition of the states, that is, the separable state $(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)/2$, the correlation difference cancels (or in other words, there is neither a number-sum nor a number-difference correlation), indicating that the state is separable. For multipartite states this procedure must be recursively invoked. However, the procedure just outlined is incomplete as it has a 'preferred basis' (it does not distinguish between, e.g., the states $(|0,0\rangle + |1,1\rangle)/\sqrt{2}$ and $(|0,0\rangle - |1,1\rangle)/\sqrt{2}$). By using the relative phase, which is an operator with a discrete spectrum but with a free, continuous parameter, this potential pitfall with the correlation method is solved.

We also introduce reduced density operators. Reducing over the jth subsystem we get the following reduced density operators:

$$\langle k_j|\hat{\rho}|k_j\rangle = p_{j;k_j}\hat{\rho}^{(j;k_j)}, \tag{8}$$

where the index k_j can run from 1 to N_j, $|k_j\rangle$ is the k_jth basis vector in subspace $\mathcal{H}_{\mathcal{Q}_j}$, $\hat{\rho}^{(j;k_j)}$ is a normalized (unit trace) density operator on the space $\mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_{j-1}\mathcal{Q}_{j+1}\cdots\mathcal{Q}_m$ and $p_{j;k_j}$ is a probability weight. (It is the probability of projecting the state onto the vector $|k_j\rangle$ when making a measurement using the computational basis in $\mathcal{Q}_j$.) Reducing more than one subsystem, e.g., subsystems 1, 2 and j, we get

$$\langle k_1, k_2, k_j|\hat{\rho}|k_1, k_2, k_j\rangle = p_{1;k_1,2;k_2,j;k_j}\hat{\rho}^{(1;k_1,2;k_2,j;k_j)}, \tag{9}$$

where, again, we have

$$\mathrm{Tr}(\hat{\rho}^{(1;k_1,2;k_2,j;k_j)}) = 1. \tag{10}$$

Following the definitions and the 'recipe' above, our entanglement measure for a multipartite pure state is given by

$$\begin{aligned}
\Gamma(\hat{\rho}) = \Bigg(& \mathcal{N}_2 \sum_{l_1>k_1}^{N_1}\sum_{k_1=1}^{N_1}\sum_{l_2>k_2}^{N_2}\sum_{k_2=1}^{N_2}\sum_{k_3=1}^{N_3}\cdots\sum_{k_m=1}^{N_m} p_{3;k_3,\ldots,m;k_m}|\mathrm{P}_2|\rho^{(3;k_3,\ldots,m;k_m)}_{\Pi(k_1,l_1,k_2,l_2)}\|^2 + \cdots + \mathcal{N}_2 \sum_{l_{m-1}>k_{m-1}}^{N_{m-1}} \\
& \times \sum_{k_{m-1}=1}^{N_{m-1}}\sum_{l_m>k_m\mathcal{N}_2}^{N_m}\sum_{k_m=1}^{N_m}\sum_{k_1=1}^{N_1}\cdots\sum_{k_{m-2}=1}^{N_{m-2}} p_{1;k_1,\ldots,m-2;k_{m-2}}|\mathrm{P}_m|\rho^{(1;k_1,\ldots,m-2;k_{m-2})}_{\Pi(k_{m-1},l_{m-1},k_m,l_m)}\|^2 \\
& + \mathcal{N}_3 \sum_{l_1>k_1}^{N_1}\sum_{k_1=1}^{N_1}\sum_{l_2>k_2}^{N_2}\sum_{k_2=1}^{N_2}\sum_{l_3>k_3}^{N_3}\sum_{k_3=1}^{N_3}\sum_{k_4=1}^{N_4}\cdots\sum_{k_m=1}^{N_m} p_{4;k_4,\ldots,m;k_m} \\
& \times |\mathrm{P}_2|\mathrm{P}_3|\rho^{(4;k_4,\ldots,m;k_m)}_{\Pi(k_1,l_1,k_2,l_2,k_3,l_3)}\|^2| + \cdots + \mathcal{N}_3 \sum_{l_{m-2}>k_{m-2}}^{N_{m-2}}\sum_{k_{m-2}=1}^{N_{m-2}}\cdots\sum_{l_m>k_m}^{N_m}\sum_{k_m=1}^{N_m} \\
& \times \sum_{k_1=1}^{N_1}\cdots\sum_{k_{m-3}=1}^{N_{m-3}} p_{1;k_1,\ldots,m-3;k_{m-3}}|\mathrm{P}_{m-1}|\mathrm{P}_m|\rho^{(1;k_1,\ldots,m-3;k_{m-3})}_{\Pi(k_{m-2},l_{l-2},k_{m-1},l_{m-1},k_m,l_m)}\|^2| \\
& + \cdots + \mathcal{N}_{m-1}\sum_{l_1>k_1}^{N_1}\sum_{k_1=1}^{N_1}\cdots\sum_{l_{m-1}>k_{m-1}}^{N_{m-1}}\sum_{k_{m-1}=1}^{N_{m-1}}\sum_{k_m=1}^{N_m} p_{m;k_m} \\
& \times |\mathrm{P}_2|\mathrm{P}_3|\cdots|\mathrm{P}_{m-1}|\rho^{(m;k_m)}_{\Pi(k_1,l_1,k_2,l_2,\ldots,k_{m-2},l_{m-2},k_{m-1},l_{m-1})}\|^2\cdots\||| \\
& + \cdots + \mathcal{N}_{m-1}\sum_{l_2>k_2}^{N_2}\sum_{k_2=1}^{N_2}\cdots\sum_{l_m>k_m}^{N_m}\sum_{k_m=1}^{N_m}\sum_{k_1=1}^{N_1} p_{1;k_1} \\
& \times |\mathrm{P}_3|\mathrm{P}_4|\cdots|\mathrm{P}_m|\rho^{(1;k_1)}_{\Pi(k_2,l_2,k_3,l_3,\ldots,k_{m-2},l_{m-2},k_{m-1},l_{m-1},k_m,l_m)}\|^2\cdots\||| + \mathcal{N}_m\sum_{l_1>k_1}^{N_1} \\
& \times \sum_{k_1=1}^{N_1}\cdots\sum_{l_m>k_m}^{N_m}\sum_{k_m=1}^{N_m}|\mathrm{P}_2|\mathrm{P}_3|\cdots|\mathrm{P}_m|\rho_{\Pi(k_1,l_1,k_2,l_2,\ldots,k_{m-1},l_{m-1},k_m,l_m)}\|^2\cdots\|| \Bigg)^{\frac{1}{2}}.
\end{aligned} \tag{11}$$

This is our central equation. It looks messy, but has a rather logical inner structure. The factors $\mathcal{N}_u$ are normalization factors, and they should not be confused with the space dimensions N_u. The first sums, where two of them are written explicitly (with normalization

factors $\mathcal{N}_2$) on the right-hand side of (11), only contribute to the respective subsystem's bipartite entanglement. There are $\binom{m}{2} = m(m-1)/2$ ways to select two systems out of m without respect to ordering. The two terms explicitly written above sum the bipartite entanglement contribution between systems $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_{m-1}$, $\mathcal{Q}_m$, respectively. For the systems $\{\mathcal{Q}_u, \mathcal{Q}_v\}$, there are $N_u(N_u-1)N_v(N_v-1)/4$ ways to select each of the relative phases of systems $\mathcal{Q}_u$ and $\mathcal{Q}_v$. Because the other system's coefficients can be chosen arbitrarily among the diagonals, there are $\Pi_{j=1}^{m} N_j/(N_u N_v)$ number of relative-phase sums and differences involving k_u, l_u, k_v and l_v each involving a different reduced density operator and an associated probability. Our permutation operator subtracts the relative-phase difference from the relative-phase sum, so by including all bipartite combinations, the bipartite entanglement of the joint system is taken care of. Next, we add the tripartite entanglement (contained in the sums with normalization factors $\mathcal{N}_3$). There are $\binom{m}{3}$ tripartite combinations, and for every choice $\{\mathcal{Q}_u, \mathcal{Q}_v, \mathcal{Q}_w\}$, where $u < v < w$, there are $N_u(N_u-1)N_v(N_v-1)N_w(N_w-1)/8$ combination of system relative phases. For each combination, we can sum all three relative phases, sum the first two and subtract the third, etc. To form differences of all combinations, we use both the permutation operators P_v and P_w. Hence, we get $4 = 2^{3-1}$ contributions within the outermost absolute signs in the sums premultiplied by $\mathcal{N}_3$ in (11). For each choice, the other system's indices can be chosen in $\Pi_{j=1}^{m} N_j/(N_u N_v N_w)$ different ways and we get the same number of reduced density operators and probabilities. For the four-partite contribution we proceed in the same way. For every choice $\{\mathcal{Q}_u, \mathcal{Q}_v, \mathcal{Q}_w, \mathcal{Q}_z\}$, where $u < v < w < z$, we use the permutation operators P_v, P_w and P_z. We get $8 = 2^{4-1}$ contributions inside the corresponding outermost absolute signs, each contribution operating on $\Pi_{j=1}^{m} N_j/(N_u N_v N_w N_z)$ different reduced density operators. The sum proceeds in this fashion until the m-partite entanglement contributions are to be added. There is only one way $\left(\binom{m}{m} = 1\right)$ to chose all subsystems, so in this case we do not reduce the density operator but use m as the index in our permutation operator. We use the permutation operators $\mathrm{P}_2, \mathrm{P}_3, \ldots, \mathrm{P}_m$. (We do not permute k_1 and l_1.) In all, we get 2^{m-1} terms inside the outermost absolute signs of the last sum in (11). These terms represent all the possible relative-phase sums and differences between all the m-systems, so there are no further terms.

From our definitions, it is clear that for any product state

$$\rho_{\Pi(k_1,l_1,k_2,l_2,\ldots,k_m,l_m)} = \rho_{k_1,l_1}\rho_{k_2,l_2}\cdots\rho_{k_m,l_m}, \tag{12}$$

where ρ_{k_u,l_u} is the indicated density matrix coefficient of system u. In this case, one gets $\mathrm{P}_u|\rho_{\Pi(k_1,l_1,\ldots,k_m,l_m)}| = 0$ for any u and any set of indices $k_1, l_1, \ldots, k_m, l_m$. Hence, our entanglement function $\Gamma(\hat{\rho}) = 0$ for any tensor product of m density operators. For entangled states, the function Γ is not invariant to local unitary transformations as will be shown explicitly below (equations (17) and (20)). Therefore, in analogy with our definitions for bipartite states, we define our measure of entanglement Γ_{sup}, where sup refers to the supremum of Γ under all possible local unitary transformations.

Note that our measure sums all the state's entanglement, all the bipartite terms, all the tripartite terms.... For bipartite states, there is only one kind of entanglement so in this particular case there is no summation to be done. For multipartite states (say, an m-partite state) there are already different possibilities to share bipartite entanglement between the subsystems, one for each choice of two subsystems out of the m. In general there are also different ways to share tripartite entanglement, etc. The only unique entanglement is the m-partite entanglement. Therefore, one should note that although, e.g., a state's bipartite entanglement between subsystems $\mathcal{Q}_1$ and $\mathcal{Q}_2$ cannot be used simultaneously either with its bipartite entanglement between subsystems $\mathcal{Q}_1$ and $\mathcal{Q}_3$, or, e.g., with its tripartite entanglement between subsystems $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$, all contributions are added in our measure. That is, our

measure characterizes the entanglement contained in a state, but in general the measure exceeds the 'usable' entanglement. However, by looking at the various terms in the sum, the usable entanglement can be extracted from the measure as it is composed of sub-sums containing the bipartite $\mathcal{Q}_1$ and $\mathcal{Q}_2$ entanglements, the bipartite $\mathcal{Q}_1$ and $\mathcal{Q}_3$ entanglements, the tripartite $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ entanglements, etc, as can be explicitly seen in (14). Actually, our measure suggests that it may be difficult to define the usable entanglement in a unique way, as, e.g., tripartite entanglement can be converted to bipartite entanglement and vice versa by local operations and classical communication. Unless the bipartite-to-tripartite entanglement conversion efficiency is independent of the state, it is not possible to relate the 'values' of the bipartite and tripartite entanglements to each other. This is in contrast to, e.g., entanglement of formation, that is defined as the minimum number of EPR-pairs needed to form a state and therefore gives a well-defined number.

Also note that our measure sums the possible cooperative entanglement. That is, it is assumed that the entanglement contained in the state is used in an optimal way. In contrast, if some subsystems are ignored, or the information contained in a subsystem is lost, the ensuing state's entanglement is in general lower than what our measure predicts. We shall come back to this point when we discuss tripartite states below.

Let us now write and use (11) in a few explicit cases. The degree of entanglement for a $\mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2}$ bipartite state is given by truncating (11) to

$$\Gamma(\hat{\rho}) = \left(\mathcal{N}_2 \sum_{l_1>k_1}^{N_1} \sum_{k_1=1}^{N_1} \sum_{l_2>k_2}^{N_2} \sum_{k_2=1}^{N_2} \left\| \rho_{(k_1-1)N_2+k_2,(l_1-1)N_2+l_2} \right| - \left| \rho_{(k_1-1)N_2+l_2,(l_1-1)N_2+k_2} \right\|^2_{\mathcal{Q}_1\mathcal{Q}_2} \right)^{\frac{1}{2}}. \tag{13}$$

This special case has already been discussed in detail in [21, 22], and we have shown that the equation coincides with the concurrence [5] for pure bipartite states in space dimension $2 \otimes 2$ (provided that one sets $\mathcal{N}_2 = 4$) and with generalized concurrence measures in $2 \otimes 3$ dimensions [6–8].

3. Tripartite entanglement

The degree of entanglement for a $\mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \mathcal{H}_{\mathcal{Q}_3}$ tripartite state is given by inserting the proper summation limits in (11):

$$\begin{aligned}
\Gamma(\hat{\rho}) = \Bigg(\mathcal{N}_2 \Bigg[& \sum_{l_1>k_1}^{N_1} \sum_{k_1=1}^{N_1} \sum_{l_2>k_2}^{N_2} \sum_{k_2=1}^{N_2} \sum_{k_3=1}^{N_3} p_{3;k_3} \left\| \rho^{(3;k_3)}_{\Pi(k_1,l_1,k_2,l_2)} \right| - \left| \rho^{(3;k_3)}_{\Pi(k_1,l_1,l_2,k_2)} \right\|^2_{\mathcal{Q}_1\mathcal{Q}_2} \\
& + \sum_{l_1>k_1}^{N_1} \sum_{k_1=1}^{N_1} \sum_{l_3>k_3}^{N_3} \sum_{k_3=1}^{N_3} \sum_{k_2=1}^{N_2} p_{2;k_2} \left\| \rho^{(2;k_2)}_{\Pi(k_1,l_1,k_3,l_3)} \right| - \left| \rho^{(2;k_2)}_{\Pi(k_1,l_1,l_3,k_3)} \right\|^2_{\mathcal{Q}_1\mathcal{Q}_3} \\
& + \sum_{l_2>k_2}^{N_2} \sum_{k_2=1}^{N_2} \sum_{l_3>k_3}^{N_3} \sum_{k_3=1}^{N_3} \sum_{k_1=1}^{N_1} p_{1;k_1} \left\| \rho^{(1;k_1)}_{\Pi(k_2,l_2,k_3,l_3)} \right| - \left| \rho^{(1;k_1)}_{\Pi(k_2,l_2,l_3,k_3)} \right\|^2_{\mathcal{Q}_2\mathcal{Q}_3} \Bigg] \\
& + \mathcal{N}_3 \sum_{l_1>k_1}^{N_1} \sum_{k_1=1}^{N_1} \sum_{l_2>k_2}^{N_2} \sum_{k_2=1}^{N_2} \sum_{l_3>k_3}^{N_3} \sum_{k_3=1}^{N_3} \Big\{ \Big\| \left| \rho_{\Pi(k_1,l_1,k_2,l_2,k_3,l_3)} \right| - \left| \rho_{\Pi(k_1,l_1,k_2,l_2,l_3,k_3)} \right\|^2_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3} \\
& - \left\| \rho_{\Pi(k_1,l_1,l_2,k_2,k_3,l_3)} \right| - \left| \rho_{\Pi(k_1,l_1,l_2,k_2,l_3,k_3)} \right\|^2_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3} \Big| \Big\} \Bigg)^{\frac{1}{2}}.
\end{aligned} \tag{14}$$

Let us now give two concrete examples of this measure for three-qubit states. In this space, equation (11) is further reduced to

$$\begin{aligned}\Gamma(\hat{\rho}) = \big(\mathcal{N}_2\big\{ & p_{3;1}\big\|\rho_{1,4}^{(3;1)}\big| - \big|\rho_{2,3}^{(3;1)}\big\|^2_{\mathcal{Q}_1\mathcal{Q}_2} + p_{3;2}\big\|\rho_{1,4}^{(3;2)}\big| - \big|\rho_{2,3}^{(3;2)}\big\|^2_{\mathcal{Q}_1\mathcal{Q}_2} \\ & + p_{2;1}\big\|\rho_{1,4}^{(2;1)}\big| - \big|\rho_{2,3}^{(2;1)}\big\|^2_{\mathcal{Q}_2\mathcal{Q}_3} + p_{2;2}\big\|\rho_{1,4}^{(2;2)}\big| - \big|\rho_{2,3}^{(2;2)}\big\|^2_{\mathcal{Q}_2\mathcal{Q}_3} \\ & + p_{1;1}\big\|\rho_{1,4}^{(1;1)}\big| - \big|\rho_{2,3}^{(1;1)}\big\|^2_{\mathcal{Q}_1\mathcal{Q}_3} + p_{1;2}\big\|\rho_{1,4}^{(1;2)}\big| - \big|\rho_{2,3}^{(1;2)}\big\|^2_{\mathcal{Q}_1\mathcal{Q}_3}\big\} \\ & + \mathcal{N}_3\||\rho_{1,8}| - |\rho_{2,7}\|^2_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3} - \|\rho_{3,6}| - |\rho_{4,5}\|^2|_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3}\big)^{1/2}. \end{aligned} \tag{15}$$

In the equation we have (somewhat superfluously) used the subscripts $\mathcal{Q}_1\mathcal{Q}_2$ etc to indicate between which subspaces the entanglement is shared.

In the three-qubit space there exist two classes of states, inequivalent under local operations and classical communication (LOCC), called $|\Psi_{\mathrm{W}}\rangle$ and $|\Psi_{\mathrm{GHZ}}\rangle$ states. They are, e.g., $|\Psi_{\mathrm{GHZ}}\rangle = (|0,0,0\rangle + |1,1,1\rangle)/\sqrt{2}$ and $|\Psi_{\mathrm{W}}\rangle = (|0,0,1\rangle + |0,1,0\rangle + |1,0,0\rangle)/\sqrt{3}$. However, let us first consider the state

$$|\Psi_{\mathrm{W}'}\rangle = \tfrac{1}{\sqrt{6}}(|0,0,0\rangle + |0,0,1\rangle + |0,1,0\rangle - |1,0,0\rangle + |1,0,1\rangle + |1,1,0\rangle). \tag{16}$$

Applying our measure directly to this state yields

$$\Gamma(\hat{\rho}_{\mathrm{W}'}) = \big(\mathcal{N}_2\big\{p_{1;1}\big|\rho_{2,3}^{(1;1)}\big|^2_{\mathcal{Q}_2\mathcal{Q}_3} + p_{1;2}\big|\rho_{2,3}^{(1;2)}\big|^2_{\mathcal{Q}_2\mathcal{Q}_3}\big\}\big)^{1/2} = \sqrt{\frac{\mathcal{N}_2}{9}}, \tag{17}$$

where we have only written the contributing terms. However, the state above is actually a $|\Psi_{\mathrm{W}}\rangle$-state in disguise. Applying a Hadamard transformation only to subsystem 1, that is, transforming the state according to $H \otimes 1 \otimes 1|\Psi_{\mathrm{W}'}\rangle$, where

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{18}$$

we get the state $|\Psi_{\mathrm{W}}\rangle$. One can show that the supremum of $\Gamma(\hat{\rho}_{\mathrm{W}'})$ is obtained by this particular unitary transformation so applying (15) to the state $H \otimes 1 \otimes 1|\Psi_{\mathrm{W}'}\rangle = |\Psi_{\mathrm{W}}\rangle$ we get

$$\begin{aligned}\Gamma_{\sup}(\hat{\rho}_{\mathrm{W}'}) = \Gamma(\hat{\rho}_{\mathrm{W}}) &= \big(\mathcal{N}_2\big\{p_{3;1}\big|\rho_{2,3}^{(3;1)}\big|^2_{\mathcal{Q}_1\mathcal{Q}_2} + p_{1;1}\big|\rho_{2,3}^{(1;1)}\big|^2_{\mathcal{Q}_2\mathcal{Q}_3} + p_{2;1}\big|\rho_{2,3}^{(2;1)}\big|^2_{\mathcal{Q}_1\mathcal{Q}_3}\big\}\big)^{1/2} \\ &= \left(\mathcal{N}_2\left\{\frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right\}\right)^{1/2} = \sqrt{\frac{\mathcal{N}_2}{2}}. \end{aligned} \tag{19}$$

This example shows that the bipartite entanglement of this state is equally shared between the three subsystems. We also see that Γ is not invariant under local unitary operations, and in more general terms, it is not invariant under local operations and classical communication. This is why we need to find the supremum of Γ under such transformations. The entanglement of the W-state is greater than that of a EPR-pair, that has $\Gamma_{\sup}(\hat{\rho}_{\mathrm{EPR}}) = (\mathcal{N}_2/4)^{1/2}$. The difference can readily be explained. If one of the qubit subsystems of the W-state is measured, then the remaining state will be in the separable state $|0,0\rangle$ with probability $1/3$, and will be in the EPR-state $(|0,1\rangle + |1,0\rangle)/\sqrt{2}$ with probability $2/3$. Since this is true irrespective of which of the three subsystems is measured, the maximum of the state's total entanglement is $(3 \times 2/3 \times \mathcal{N}_2/4)^{1/2} = (\mathcal{N}_2/2)^{1/2}$.

Let us now turn to the GHZ-state. From this example we shall see that finding the supremum of Γ is associated with fundamental difficulties (and not only with computational difficulties). Computing Γ for the $|\Psi_{\mathrm{GHZ}}\rangle$-state we get

$$\Gamma(\hat{\rho}_{\mathrm{GHZ}}) = \big(\mathcal{N}_3|\rho_{1,8}|^2_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3}\big)^{\frac{1}{2}} = \sqrt{\frac{\mathcal{N}_3}{4}}. \tag{20}$$

Hence, the GHZ-state contains no bipartite entanglement *per se*. However, if we make the following unitary transformation:

$$|\Psi'_{\mathrm{GHZ}}\rangle = H \otimes H \otimes H |\Psi_{\mathrm{GHZ}}\rangle \\ = \tfrac{1}{2}(|0,0,0\rangle + |0,1,1\rangle + |1,0,1\rangle + |1,1,0\rangle), \tag{21}$$

we get

$$\Gamma(\hat{\rho}'_{\mathrm{GHZ}}) = \sqrt{\frac{3\mathcal{N}_2}{4}}. \tag{22}$$

Transforming the $|\Psi_{\mathrm{GHZ}}\rangle$-state in this manner transforms the tripartite entanglement to bipartite entanglement. The fact that this is possible should not come as a surprise, given that the $|\Psi_{\mathrm{GHZ}}\rangle$-state can be formed making local operations on 2 EPR-pairs shared by three parties [23]. Note that in order to use the potential bipartite entanglement, the parties holding the three subsystems must cooperate. If any one of the subsystems is traced out, the ensuing bipartite state is separable. In contrast, if the qubit value of one of the subsystems is measured and communicated to one of the parties holding either of the other subsystems, the remaining bipartite state will be in a known EPR-state. If such a measurement is done directly on the untransformed GHZ-state, all entanglement is lost regardless of whether or not the outcome is communicated to the other parties. It is easy to understand that by applying a proper, local qubit transformation and subsequently measuring the qubit and communicating the result, it is possible to continuously vary the ensuing bipartite entanglement from zero to that of an EPR-state.

We also see that the bipartite entanglement of a GHZ-state actually is higher than that of a W-state, provided that the parties cooperate. In the case of the GHZ-state, the parties will always end up with a known EPR-state if one the subsystems is measured and the three parties cooperate. In the case of the W-state, this only happens with a probability $2/3$. In addition, it becomes clear that before assigning relative values to $\mathcal{N}_2$ and $\mathcal{N}_3$, it is not possible to establish the supremum of Γ for a GHZ-state because it depends on the ratio between $\mathcal{N}_2$ and $\mathcal{N}_3$. It is, however, possible to do what we have done, namely establish the supremum for the bipartite entanglement and for the tripartite entanglement separately. Because the GHZ- and the W-states belong to different equivalence classes, their relative bipartite and tripartite entanglement weights are not obvious. This issue is related to the, still open, question about minimal reversible entanglement generating sets [2, 12, 11, 19, 24].

4. Beyond three-partite qubit entanglement

Next, let us look at an interesting four-qubit state $|\Psi_1\rangle = (|0,0,0,1\rangle + |0,1,0,0\rangle + |1,0,1,0\rangle + |1,1,1,1\rangle)/2$. Applying our measure of entanglement directly to this state we get

$$\begin{aligned}\Gamma\big(\hat{\rho}_{\Psi_1}\big) = \big(&\mathcal{N}_2\big(p_{1;1,3;1}\big|\rho_{2,5}^{(1;1,3;1)}\big|^2_{\mathcal{Q}_2\mathcal{Q}_3} + p_{1;2,3;2}\big|\rho_{11,16}^{(1;2,3;2)}\big|^2_{\mathcal{Q}_2\mathcal{Q}_4}\big) \\ &+\mathcal{N}_3\big(p_{4;2}\big|\rho_{1,8}^{(4;2)}\big|^2_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3} + p_{4;1}\big|\rho_{3,6}^{(4;1)}\big|^2_{\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3} \\ &+ p_{2;1}\big|\rho_{2,7}^{(2;1)}\big|^2_{\mathcal{Q}_1\mathcal{Q}_3\mathcal{Q}_4} + p_{2;2}\big|\rho_{1,8}^{(2;2)}\big|^2_{\mathcal{Q}_1\mathcal{Q}_3\mathcal{Q}_4}\big)\big)^{\frac{1}{2}} \\ = &\left(\frac{\mathcal{N}_2}{4} + \frac{\mathcal{N}_3}{2}\right)^{\frac{1}{2}}.\end{aligned} \tag{23}$$

Again, establishing a supremum requires the ratio between the two normalization coefficients to be fixed. The state has both bipartite and tripartite entanglements. In this case too, the parties possessing the different qubit subsystems must cooperate in order to use the bipartite

entanglement. If, e.g., qubits 1 and 3 are measured in the computational basis, the result is either two zeros or two ones. If this result is communicated to the parties holding qubits 2 and 4 (that is, we perform a LOCC, optimal for bringing out the bipartite entanglement), the remaining two-qubit state will be in (a known) pure EPR-state. If, on the other hand, we measure only qubit 2 or qubit 4 and communicate the result, then the remaining state is in a known GHZ-state.

The tripartite entanglement of the state $|\Psi_2\rangle = (|0,1,1,0\rangle + |1,0,0,1\rangle + |0,1,1,1\rangle + |1,0,0,0\rangle)/2$, on the other hand, is given by

$$\Gamma_{\sup}\big(\hat{\rho}_{\Psi_2}\big) = \sqrt{\frac{\mathcal{N}_3}{4}}. \tag{24}$$

That is, the state has no four-partite entanglement and as before, the tripartite entanglement can be converted to bipartite entanglement. (The supremum in the equation refers only to the tripartite entanglement.) To arrive at the result, we note that a unitary transformation $\hat{U}_4$ local to $\mathcal{Q}_4$ can transform the state into, e.g., $\hat{U}_4|\Psi_2\rangle = (|0,1,1\rangle + |1,0,0\rangle \otimes |0\rangle)/\sqrt{2}$ for which one finds the supremum of the tripartite part of Γ. In this case, the state's tripartite entanglement is the same whether or not the party in possession of qubit 4 cooperates or not.

As a last example, consider the m-qubit generalization of a GHZ-state, a so-called m-GHZ state:

$$|\Psi_{m\text{-GHZ}}\rangle = \tfrac{1}{\sqrt{2}}\left(|0_1, 0_2, \ldots, 0_m\rangle + |1_1, 1_2, \ldots, 1_m\rangle\right). \tag{25}$$

Applying our measure of entanglement to this state and maximizing, we get

$$\Gamma_{\sup}(\hat{\rho}_{m\text{-GHZ}}) = \left(\frac{\mathcal{N}_m}{4}\right)^{\frac{1}{2}}, \tag{26}$$

where the supremum refers to the m-partite entanglement. In analogy with the tripartite GHZ-state, it is possible to trade the m-partite entanglement to entanglement between fewer than m subsystems, provided that the various parties holding the subsystems cooperate.

5. Conclusions

In conclusion, we have proposed an entanglement measure for pure, multipartite quantum states. The measure directly detects product states (it is zero for such states), and it quantifies the entanglement of any pure state up to the bipartite, tripartite, ..., m-partite normalization coefficients. Since it is not possible to convert the entanglement in states with incompatible entanglement classes such as GHZ- and W-states into each other, it may not be meaningful to specify the coefficients relative to each other. Rather, from an operational point of view, it seems more meaningful to specify each shared entanglement separately, e.g., in a system composed of four subsystems $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and $\mathcal{Q}_4$, it is meaningful to discuss, separately, the bipartite entanglement between, e.g., systems $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and $\mathcal{Q}_1$ and $\mathcal{Q}_4$. While it is meaningful to relate the resources needed to form various kinds of entangled states in terms of the minimum number of EPR-pairs needed, it is not yet clear if the utility of entanglement in applications can be classified in a similar, unambiguous manner. That would require a proof that various kinds of entanglement, e.g., bipartite and four-partite entanglements can be converted into each other with a fixed 'efficiency' regardless of the state. Such a proof is lacking at present, and to us it seems unlikely that this could be the case in view of how many kinds, and how many combinations, of different entanglements a state can have simultaneously. (It also seems likely that other states than EPR-pairs would be efficient, or needed, to convert between different kinds of entanglements.) It seems more likely that entanglement will have

to be classified in more than one way, depending on what kind of information one seeks and what kind of application one has in mind. Each measure will then have its own meaning and merit. Our measure sums all contributions to quantify the state's entire entanglement, but, as just indicated, from an operational viewpoint, it is rather the sum's various contributions that have a well-defined meaning.

Acknowledgments

We wish to thank Professor L L Sánchez-Soto for inspiration and useful discussions. This work was supported by the Swedish Research Council (VR) and the Swedish Foundation for Strategic Research (SSF).

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